Permutation & Combination

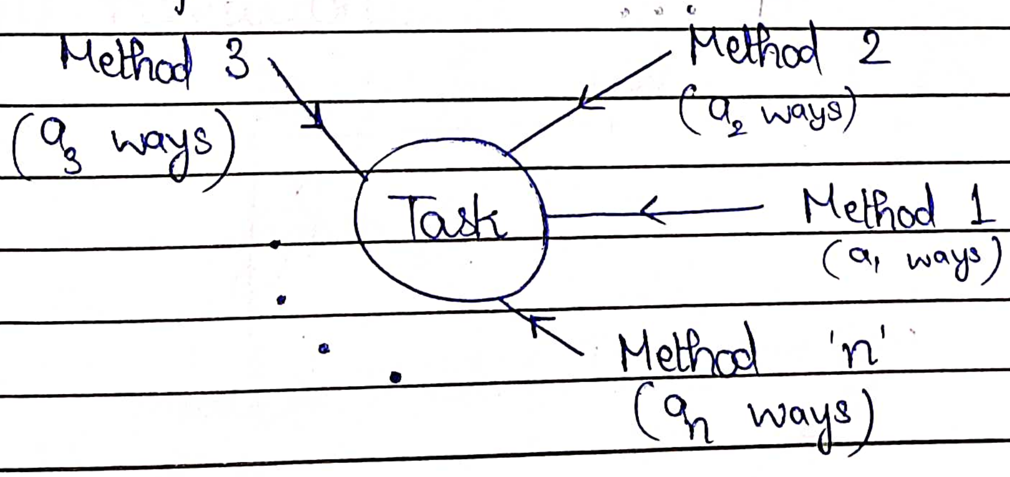
Principles of Counting –

1) Rule of Multiplication –

Steps:	1st	2nd	...	nth	→ Task
#Ways:	a_1	a_2	...	a_n	

$$\boxed{\text{Total Ways} = a_1 a_2 \ldots a_n \text{ (to complete task)}}$$

2) Rule of Add^n –



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$$\boxed{\underset{\text{(to complete task)}}{\text{Total Ways}} = (a_1 + a_2 + \dots + a_n)}$$

3) Bijection Principle

Let say $\exists$ 2 sets A & B. Let no. of elements in B be 'n'.

If we show that $\exists$ a ~~bije~~ bijection from A to B $\Rightarrow$ $\exists$ n elements in A.

4) Principle of Inclusion & Exclusion —

$$\boxed{n\left(\bigcup A_i\right) = \sum n(A_i) - \sum n(A_i \cap A_j) + \sum n(A_i \cap A_j \cap A_k) - \sum n(A_i \cap A_j \cap A_k \cap A_l) + \dots}$$

Permutation & Combination

Permutation — Arrangments (Order ✓)

Combination — Selection (Order X)

ABC (Taking 2 at a time)

Permutation	
AB	BC
AC	CA
BA	CB

Combination
AB
AC
BC

Permutation

'n' DISTINCT objs. available

Arrangments of size 'r'. $(r \leq n)$

1) W/O repetition

$$\boxed{\# \text{ways} = \left(\frac{\underline{|n}}{\underline{|n-r}}\right) = {}^{n}P_{r}}$$

<u>Proof:</u> Situation is equiv. to arranging 'r' ppl. out of 'n' which can sit at 'r' possible places.

Place:	1	2	3	...	r
Ways:	n	$(n-1)$	$(n-2)$		$(n-r+1)$

Total arrangments $= n(n-1)\ldots(n-r+1)$

$$\Rightarrow \boxed{\text{Total} = \frac{|\underline{n}}{|\underline{n-r}}}$$

2) <u>With repetition</u> —

$$\boxed{\#\text{ways} = n^r}$$

<u>Proof:</u> Situation is equiv. to 'n' stamps which have 'r' places to be printed.

Places:	1	2	...	r
Ways:	n	n		n

$$\boxed{\text{Total} = n^r}$$

Special Case :— No. of arrangments that can be formed using 'n' objs. out of which 'p' are identical (it ~~con~~ of one kind), then 'q' are identical (it of 2nd kind) it 'r' are identical (it of 3rd kind) it rest are diff., then

$$\boxed{\# \text{ arrangments} = \left(\frac{\lfloor n}{\lfloor p \lfloor q \lfloor r}\right)}$$

by taking all at a time!.

Combination

'n' DISTINCT objs. are available.
Selection of size 'r' $(r \leq n)$

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1) W/O repetition —

$$\boxed{\# \text{ways} = \left(\frac{\lfloor n}{\lfloor r \lfloor n-r}\right) = {}^nC_r}$$

Proof: Assume 'x' is total no. of combinations of 'r' size.

For each combination there are $\lfloor r$ arrangements

$\Rightarrow$ (Total arrangements) $= x \cdot \lfloor r = {}^nP_r$

$$\Rightarrow \boxed{(\# \text{Combinations}) = \left(\frac{{}^nP_r}{\lfloor r}\right)}$$

2) With repetition —

(Taken later)

Circular Permutation

Arrangment along a close curve, especially a circle.

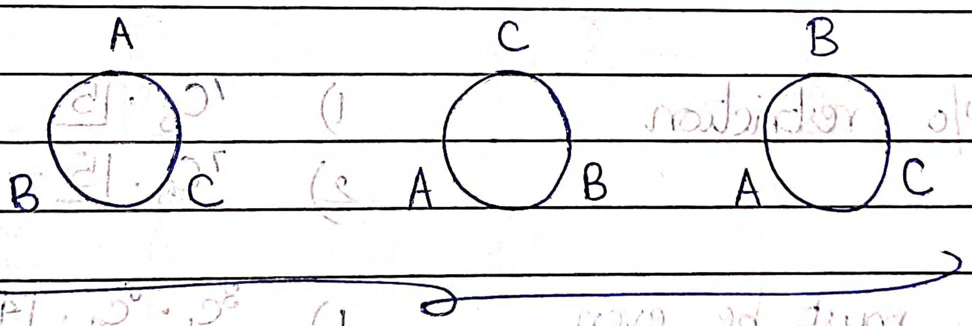
'n' DISTINCT objs. are to be arranged on 'n' places on a circle.

$$\boxed{\#\text{ways} = \overline{|n-1}}$$

Proof: Seat one ~~person~~ obj. to define the start & end of arrangment.
Then $\boxed{\#\text{ways} = \overline{|n-1}}$

If in 2 permutation relative posⁿ. of objs. is same, then the 2 permutations are identical.

Eg:

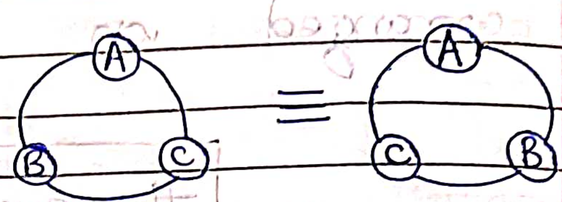


Same

If we are unable to make distinction b/w ~~anti~~ ↺ & ↻ permutations, then

$$\boxed{\# \text{ways} = \left(\frac{\underline{|n-1}}{2}\right)}$$

Eg: Necklace



Q) Find 5 digit no. which can be made by using digits w/o repetition given below —

1) 1, 2, 3, 4, 5, 6, 7

2) 0, 1, 2, 3, 4, 5, 6

a) w/o restriction

1) $^7C_5 \cdot \underline{|5}$

2) $^7C_5 \cdot \underline{|5} - {}^6C_4 \cdot \underline{|4}$

b) no. must be even

1) $^3C_1 \cdot {}^6C_4 \cdot \underline{|4}$

2) $^6C_4 \cdot \underline{|4} + {}^3C_1 \cdot {}^5C_4 \cdot \underline{|4}$

c) no. must be odd

1) ${}^{4}C_1 \cdot {}^{6}C_4 \cdot \underline{|4}$

2) $\cancel{{}^{8}C_1 \cdot {}^{5}C_4 \cdot \underline{|4}}$ ${}^{3}C_1 \cdot {}^{5}C_1 \cdot {}^{5}C_3 \cdot \underline{|3}$

d) no. must be divisible by 4

1) $10 \cdot {}^{5}C_3 \cdot \underline{|3}$

2) $4 \cdot {}^{5}C_3 \cdot \underline{|3} + 8 \cdot \cancel{{}^{4}C_3 \cdot \underline{|3}}$

$\cdot 4 \cdot 4 \cdot 3$

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Restricted Selection / Arrangement

1) The # ways in which 'r' obj.s can be selected from 'n' diff. objs. if 'k' particular objs. are —

1.1) always inc. = ${}^{(n-k)}C_{(r-k)}$

1.2) never inc. = ${}^{(n-k)}C_r$

~~1.3)~~

2) The # arrangements of 'n' distinct obj.s. taken 'r' at a time s.t. 'k' particular objs. are —

2.1) always inc. = $\lfloor r \cdot {}^{(n-k)}C_{(r-k)}$

2.2) never inc. = $\lfloor r \cdot {}^{(n-k)}C_r$

Q) 1) How many Δs can be formed by joining vertices of an 'n' sided regular polyⁿ.?

2) How many of these have exactly 1 side common with polyⁿ?

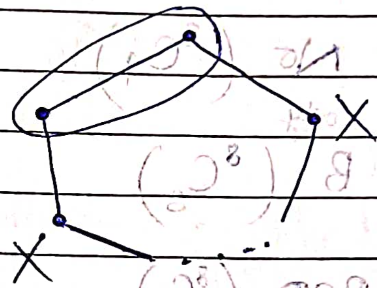
3) How many Δs have exactly 2 sides common with poly^n?

4) How many Δs have no sides common with poly^n?

A) 1) ${}^nC_3$

2) i) Select a pair of adj. vertices i.e. a side

ii) Select final vertex from $(n-4)$



$\#$ways $= n \cdot {}^{(n-4)}C_1$

3) n (Select 1 vertex, select both adj. vertices)

4) ${}^nC_3 - n \cdot {}^{(n-4)}C_1 - n$

Combination with Repetition

n distinct objs : $A_1, A_2, \ldots, A_n$

Size $=$ r Selection, rep. allowed.

Let A_1 come x_1 times, A_2 come x_2 times, $\ldots$, A_n come x_n times.

Since size $= r \Rightarrow$ $\boxed{\sum_{i=1}^{n} (x_i) = r}$ — ①

Since obj. may or may NOT be selected,

$\boxed{0 \leqslant x_i \leqslant r}$ $\forall i \in \{1, \ldots, n\}$ — ②

Now, ~~by Bijective Principle~~, it can be observed that

$$\boxed{\begin{pmatrix}\text{No. of int.} \\ \text{sol}^n\text{s of ①}\end{pmatrix} = \begin{pmatrix}\text{No. of selections} \\ \text{with rep. of size 'r'}\end{pmatrix}}$$

Using Multinomial Theorem,

$$\begin{pmatrix}\text{No. of int} \\ \text{sol}^n\text{s of ①}\end{pmatrix} = \begin{pmatrix}\text{Coeff. of } t^r \\ \text{in} \\ (1+t+\ldots t^r)^n \\ \text{OR} \\ (1+t+t^2+\ldots)^n\end{pmatrix} = \begin{pmatrix}\text{Coeff of } t^r \\ \text{in} \\ (1-t)^{-n}\end{pmatrix} = {}^{n+r-1}C_{n-1}$$

☆ (as any powers greater than r do NOT matter)

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Multinomial Theorem

Let $x_1, x_2, \dots, x_m$ be integers &

let ① — $\boxed{x_1 + x_2 + \dots + x_m = n}$ where

$a_1 \leqslant x_1 \leqslant b_1$, $a_2 \leqslant x_2 \leqslant b_2$, ..., $a_m \leqslant x_m \leqslant b_m$

$$\boxed{\begin{pmatrix}\text{Total no. of int. sol}^n \\ \text{of eq}^n \text{ ①}\end{pmatrix} = \begin{pmatrix}\text{Coeff. of } t^n \\ \text{in exp}^n \text{ ②}\end{pmatrix}}$$

Now,

② — $\boxed{\left(t^{a_1}+t^{a_1+1}+\dots+t^{b_1}\right)\left(t^{a_2}+t^{a_2+1}+\dots+t^{b_2}\right)\dots\left(t^{a_m}+t^{a_m+1}+\dots+t^{b_m}\right)}$

☆ In combination with repetition & multinomial theorem, formula is NOT imp., ~~me~~ approach is imp!

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Q) find the total no. of ways of selecting 10 balls out of unlimited no. white, red, blue & green balls.

Q) Find no. of ways in which an examiner can assign 30 marks to 8 Qs, given not less than 2 marks to any Q. (Marks given in integers only)

A) Coeff. of t^{10} in $(1+t+t^2+\dots)^4$

$\Rightarrow$ " " t^{10} " $(1-t)^{-4} = \left({}^{4+10-1}C_{10}\right) = {}^{13}C_3$

$\{t_w + t_r + t_b + t_g = 10\}$

A) Coeff. of t^{30} in $\left(t^2+t^3+\dots+t^{\cancel{30}\,16}\right)^8$

$\Rightarrow$ " " t^{30} in $\left(t^2+t^3+\dots\right)^8$

$\Rightarrow$ " " t^{30} in $\dfrac{t^{16}}{(1-t)^8}$

$\Rightarrow$ " " $t^{\cancel{24}\,14}$ in $(1-t)^{-8} = {}^{\cancel{24}\,14+8-1}C_{\cancel{24}\,14} = {}^{21}C_7$

$\{t_1 + t_2 + \dots + t_8 = 30, \quad t_i \geq 2\}$

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☆ Q) Let 15 toys be distributed among 3 children subject to the condⁿ that any child can take any no. of toys. Find req. no. of ways to do this if

1) toys are distinct.

2) toys are identical.

A) 1) Each toy ~~can~~ has 3 choices for children

$\Rightarrow$ $\boxed{\#\text{ways} = 3^{15}}$

2) Let C_i get x_i toys $\Rightarrow$ $x_1 + x_2 + x_3 = 15$ $\{x_i \geq 0\}$

$\Rightarrow$ Coeff. of t^{15} in $(1+t+t^2+\ldots+t^{15})^3$

$\Rightarrow$ Coeff. of t^{15} in $(1+t+t^2+\ldots)^3$

$\Rightarrow$ Coeff. of t^{15} in $(1-t)^{-3}$ $= {}^{3+15-1}C_{3-1} = {}^{17}C_2$

☆ Q) Find total no. of 6 digits nos. $x_1 x_2 x_3 x_4 x_5 x_6$ having propᵗ that

$x_1 < x_2 \leq x_3 < x_4 < x_5 \leq x_6$.

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Observe that $x_1 \neq 0$. #ways

A) C1: $x_1 < x_2 < x_3 < x_4 < x_5 < x_6$ — ${}^{9}C_6$

C2: $x_1 < x_2 = x_3 < x_4 < x_5 < x_6$ — ${}^{9}C_5$

C3: $x_1 < x_2 < x_3 < x_4 < x_5 = x_6$ — ${}^{9}C_5$

C4: $x_1 < x_2 = x_3 < x_4 < x_5 = x_6$ — ${}^{9}C_4$

Total = 462

No need to arrange as order is already defined!

All Possible Selections

1) Selection from distinct objs —

$$\left(\begin{array}{l}\text{No. of selection from 'n' diff. objs} \\ \text{taking at least 1 } \cancel{\text{at a time}}\end{array}\right) = \left({}^{n}C_1 + {}^{n}C_2 + \dots + {}^{n}C_n\right)$$
$$= 2^n - 1$$

Alt. Exp: Each obj. has 2 choices — Select ✓ or Select X. ⇒ #ways $= 2^n$.
Since ∃ 1 case s.t. all objs. Select X ⇒ Req. $= (2^n - 1)$

2) Selection from identical objs. —

2.1) (No. of selection of 'r' objs. out of 'n' identical objs.) = 1

2.2) (Total no. of selections of ≥ 0 objs. from 'n' identical objs.) = $n+1$

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3) Selection of at least 1 out of $(a_1+a_2+\dots+a_n)$ objs. where 'a_1' are alike of ~~1st~~ 1st kind, 'a_2' are alike of 2nd kind, ... 'a_n' are alike of nth kind

$$= \boxed{[(a_1+1)(a_2+1)\dots(a_n+1) - 1]}$$

4) Selection when both identical & distinct objs. are present —

4.1) (No. of selections out of $(\underset{1^{st}\text{ kind}}{a_1} + \underset{2^{nd}\text{ kind}}{a_2} + \dots + \underset{n^{th}\text{ kind}}{a_n} + \underset{\text{distinct}}{k})$ objs.)

$$= \boxed{[(a_1+1)(a_2+1)\dots(a_n+1)(2^k) - 1]}$$

Application —

Q) Total no. of divisors of a given no. 'n' $(n \geq 2, n \in \mathbb{N})$

A) Let $\boxed{n = p_1^{\alpha_1} p_2^{\alpha_2} \dots p_m^{\alpha_m}}$ with $p_i \in \mathbb{P}$ & $\alpha_i \in \mathbb{N}$

Let 'd' be its divisor s.t. $\boxed{d = p_1^{\beta_1} p_2^{\beta_2} \dots p_m^{\beta_m}}$
$p_i \in \mathbb{P}$ & $\beta_i \leq \alpha_i$ & $\beta_i \in \mathbb{N} \cup \{0\}$

(No. of possible values of 'd') = (No. of divisors of 'n')

$\Rightarrow$ $\boxed{(\#\text{ divisors}) = (\alpha_1+1)(\alpha_2+1)\dots(\alpha_m+1)}$

Now, $\boxed{\left(\text{Sum of divisors}\right) = \left(\frac{p_1^{\alpha_1+1}-1}{p_1-1}\right)\left(\frac{p_2^{\alpha_2+1}-1}{p_2-1}\right)\dots\left(\frac{p_m^{\alpha_m+1}-1}{p_m-1}\right)}$

Q) If $n = 10800$, find $\{n = 2^4 \cdot 3^3 \cdot 5^2\}$

1) total # divisors of 'n'

A) $(4+1)(3+1)(2+1)$

2) total # proper divisors of 'n'.

A) $(4+1)(3+1)(2+1) - 1$

3) total # even divisors

A) $(3+1)(3+1)(2+1)$ {Assume ~~2~~ one of the 2s is already taken}

4) total # divisors of form $(4m+2)$.

A) $(3+1)(2+1)$ {Assume ~~one~~ only one of the 2s taken}

5) total no. of divisors which are multiple of 15

A) $(4+1)(2+1)(1+1)$ {Assume one of the 3s & one of the 5s is already taken}

6) Sum of all divisors of 'n'.

A) $(1+2^1+2^2+2^3+2^4)(1+3^1+3^2+3^3)(1+5^1+5^2)$

$$= \left(\frac{2^5-1}{2-1}\right)\left(\frac{3^4-1}{3-1}\right)\left(\frac{5^3-1}{5-1}\right)$$

7) Sum of all all proper divisors

A) $\left(\frac{2^5-1}{2-1}\right)\left(\frac{3^4-1}{3-1}\right)\left(\frac{5^3-1}{5-1}\right) - 2^4 \cdot 3^3 \cdot 5^2$

8) Sum of all even divisors.

A) $(2^1+2^2+2^3+2^4)(1+3^1+3^2+3^3)(1+5^1+5^2)$

$$= 2\left(\frac{2^4-1}{2-1}\right)\left(\frac{3^4-1}{3-1}\right)\left(\frac{5^3-1}{5-1}\right)$$

9) Sum of odd proper divisors.

A) Since n = even ⇒ {Odd proper divisors} = {Odd divisors}

$$(1+3+3^2+3^3)(1+5+5^2) = \left(\frac{3^4-1}{3-1}\right)\left(\frac{5^3-1}{5-1}\right)$$

10) sum of divisors which are divisible by 15.

A) $(1+2^1+2^2+2^3+2^4)(3^1+3^2+3^3)(5+5^2)$

$$= 15\left(\frac{2^5-1}{2-1}\right)\left(\frac{3^3-1}{3-1}\right)\left(\frac{5^2-1}{5-1}\right)$$

11) i) no. of ways in which 'n' can be resolved as a product of 2 factors.

A) Since $n \neq \square$, we have (# divisors) = even.

Observe, $d_1 d_2 = d_2 d_1 \Rightarrow$ (# Req. pairs.) $= \frac{(\text{\# divisors})}{2}$

$$\Rightarrow \text{Req.} = \frac{(4+1)(3+1)(2+1)}{2}$$

ii) Repeat 11) i) by $n = 2^2 \cdot 3^4 \cdot 5^6$.

~~A) Since $n = \square$, we have (# divisors) = odd.~~

~~Observe $d_1 d_2 = d_2 d_1 \Rightarrow$ (# Req. pairs) = (# diff. divisors pair)/2 + (# same divisors pair)~~

~~# diff. divisors $\Rightarrow$ (# Req. pairs) = (# diff. divisors pair) + (# odd div~~

A) $(\#\text{ divisors}) = (2+1)(4+1)(6+1)$

$$(\#(d_1, d_2) \text{ s.t. } d_1 \neq d_2) = \left(\frac{(2+1)(4+1)(6+1)-1}{2}\right)$$

$$(\#(d_1, d_2) \text{ s.t. } d_1 = d_2) = 1 \rightarrow \frac{(2+1)(4+1)(6+1)+1}{2}$$

{take $d_1 = d_2 = 2 \cdot 3^2 \cdot 5^3$}

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12) # ways in which 'n' can be resolved as product of 2 coprime factors.

A) A divisor 'd' has $\beta_i = \alpha_i$ for p_i or $\beta_i = 0$ for p_i

Possible Pairs $= (1, n);\ (2^4, n/2^4);\ (3^3, n/3^3);\ (5^2, n/5^2)$

$\Rightarrow$ Req $= 4$

13) # ways in which 'n' can be ~~written~~ resolved as product of 2 non-coprime factors.

A) (# Any pairs) $-$ (# Coprime pairs) $= \frac{(4+1)(3+1)(2+1)}{2} - 4$

14) no. of factors of 'n' which are perfect sq.s.

A) $n = 2^4 \cdot 3^3 \cdot 5^2 = 3 \cdot (4)^2 \cdot (25)^1$

(# $\square$ factors) $= (2+1)(1+1)$

Legendre Formula

To find highest power of prime 'p' in $\lfloor n$.

$$\boxed{v_p(n) = \left[\frac{n}{p}\right] + \left[\frac{n}{p^2}\right] + \left[\frac{n}{p^3}\right] + \dots}$$

Derangement

Any change in existing order of things is called derangement.

(# ways in which 'n' objs. kept in a row can be deranged s.t. No obj. at its correct ~~post~~ post.)

$$= \boxed{D_n = (\lfloor n)\left(1 - \frac{1}{\lfloor 1} + \frac{1}{\lfloor 2} - \frac{1}{\lfloor 3} + \dots + \frac{(-1)^n}{\lfloor n}\right)}$$

Eg: R B G Y

R B G Y

$D_n = $ (No restrict arrangements) $-$ (At least 1 ~~at~~ in correct ~~place~~ box.)

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Let 'R' event denote R is ~~at~~ in correct ~~place~~ box.

By Principle of Inclusion & Exclusion,

$$n(R\cup B\cup G\cup Y) = \left(\sum\left[n(R)\right] - \sum\left[n(R\cap B)\right] + \sum\left[n(R\cap B\cap G)\right] - \sum\left[n(R\cap B\cap G\cap Y)\right]\right)$$

$$= \left(\underbrace{{}^4C_1}_{\text{select what 1 in correct box}} \cdot \underbrace{\underline{|3}}_{\text{arrange rest}} - \underbrace{{}^4C_2}_{\text{select what 2 in correct boxes}} \cdot \underbrace{\underline{|2}}_{\text{arrange rest}} + \underbrace{{}^4C_3}_{\text{select what 3 in correct boxes}} \cdot \underbrace{\underline{|1}}_{\text{arrange rest}} - \underbrace{{}^4C_4}_{\text{select what 4 in correct boxes}} \cdot \underbrace{\underline{|0}}_{\text{arrange rest.}}\right)$$

$$= \left(\frac{\underline{|4}}{\underline{|1}} - \frac{\underline{|4}}{\underline{|2}} + \frac{\underline{|4}}{\underline{|3}} - \frac{\underline{|4}}{\underline{|4}}\right)$$

Hence, $D_4 = \underline{|4} - \left(\frac{\underline{|4}}{\underline{|1}} - \frac{\underline{|4}}{\underline{|2}} + \frac{\underline{|4}}{\underline{|3}} - \frac{\underline{|4}}{\underline{|4}}\right)$

$$\Rightarrow \quad D_4 = \underline{|4}\left(1 - \frac{1}{\underline{|1}} + \frac{1}{\underline{|2}} - \frac{1}{\underline{|3}} + \frac{1}{\underline{|4}}\right)$$

Similarly we can extend for 'n' objs.

Division & Distribution of Objs.

(with fix. objs. in each grp.)

1) Into grps. of Unequal Sizes —

1.1) (No. of ways to divide 'n' distinct objs. into 'r' grps. each containing $q_1, q_2, \ldots$ objs. $(q_i \neq q_j)$) $= \dfrac{\underline{|n}}{\underline{|q_1}\,\underline{|q_2} \cdots \underline{|q_r}}$

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1.2) $\left(\begin{array}{l}\text{No. of ways to distribute 'n' distinct objs among 'r'} \\ \text{people s.t. some gets } q_1 \text{ objs, other gets } q_2 \text{ objs., } \dots \ (q_i \neq q_j) \\ \star \text{(anyone can get any no. of objs.)}\end{array}\right) = \dfrac{\lfloor n}{\lfloor q_1 \lfloor q_2 \cdots \lfloor q_r} \cdot \lfloor r$

2) Into grps. of Equal Sizes —

2.1) $\left(\begin{array}{l}\text{No. of ways to divide 'mn' distinct objs. equally} \\ \text{into 'n' grps (i.e. unmarked grps)} \star\end{array}\right) = \dfrac{\lfloor mn}{(\lfloor m)^n \cdot \lfloor n}$

2.2) $\left(\begin{array}{l}\text{No. of ways to distribute 'mn' distinct objs. equally} \\ \text{among 'n' people (i.e. marked grps.)} \star\end{array}\right) = \dfrac{\lfloor mn}{(\lfloor m)^n}$

Q) Find no. of ways to divide 52 cards among 4 players equally.

A) $\dfrac{\lfloor 52}{(\lfloor 13)^4 \cdot \lfloor 4} \cdot \lfloor 4 = \dfrac{\lfloor 52}{(\lfloor 13)^4} = (\#\text{ways})$

Q) Find # ways to give 16 diff. things to 3 people A, B, C s.t. B gets 1 more than A & C gets 2 more than B.

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A) $\dfrac{\lfloor 16}{\lfloor 4 \cdot \lfloor 5 \cdot \lfloor 7} = (\#\text{ways})$ (No need to distribute as order already decided)

Q) In how many ways can 8 diff. books be distributed among 3 students if each receives at least 2 books.

A) Possibilities : $\{2, 2, 4\}$; $\{2, 3, 3\}$

$$(\#\text{ways}) = \left[\frac{\lfloor 8}{\lfloor 4 \cdot (\lfloor 2)^2 \cdot \lfloor 2} + \frac{\lfloor 8}{\lfloor 2 \cdot (\lfloor 3)^2 \cdot \lfloor 2} \right] \times \lfloor 3$$

1st possible division — 2nd possible division — Distribution

☆ At end we do NOT multiply by $\lfloor 3 / \lfloor 2$!
Instead we multiply by $\lfloor 3$

This is because Grps. of Equal SIZE are NOT identical !

Q) 'n' diff. toys are to be distributed among 'n' children. Find # ways in which toys can be distributed s.t. exactly 1 child gets no toy.

A) Grps : $0, \underbrace{1, \ldots, 1}_{(n-2)\ 1s}, 2$

$$(\# \text{ways}) = \underbrace{\frac{\lfloor n}{\lfloor 0 \cdot (\lfloor 1)^{(n-2)} \cdot \lfloor n-2 \cdot (\lfloor 2)}}_{\text{division}} \cdot \underbrace{\lfloor n}_{\text{distribution}} = \cancel{\frac{\lfloor n}{2}} \cancel{\left({}^{n}C_{2}\right)}$$

$${}^{n}C_{2} \cdot \lfloor n$$

Imp. Results

1) Distribution of distinct objs. when grp. size are not fix.

1.1) (Empty grps. are allowed) :

Consider distribution of 'n' distinct objs. among 'r' people when anyone can get any no. of toys.

$$= r^{n}$$

1.2) (Empty grps. are not allowed) :

Consider distr. of 'n' distinct objs. among 'r' people if each of them gets at least 1 obj.

$$= r^{n} - {}^{r}C_{1} \cdot (r-1)^{n} + {}^{r}C_{2} \cdot (r-2)^{n} \ldots + (-1)^{(r-1)}\, {}^{r}C_{(r-1)} \left(r-(r-1)\right)^{n}$$

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2) Distribution of identical objs —

2.1) (When empty grps. are not allowed):

Consider 'n' objs. to be distr. among 'r' people

$$= {}^{(n-1)}C_{(r-1)}$$

It is equiv. to # sdⁿ of

$$x_1 + x_2 + \dots + x_r = n \; , \qquad x_i \geq 1$$

2.2) (When empty grps. are allowed):

Consider 'n' objs to be distr. among 'r' people

$$= {}^{(n+r-1)}C_{(r-1)}$$

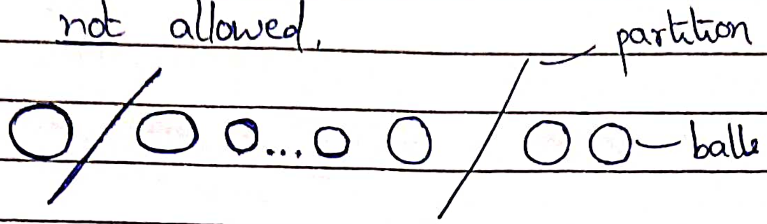
Partition Method

Alt. expⁿ for last two cases.

Consider 'n' balls to be distr. among 'r' people.

Here, we arrange the 'n' balls with '(r−1)' partitions.

C1: Empty not allowed.



Since grps. can't be empty we get (n−1) gaps. as first & last objs. are balls only.

We choose (r−1) ~~places~~ gaps. for partitions $\Rightarrow {}^{(n-1)}C_{(r-1)}$

C2: Empty allowed

Since grps. can be empty. We arrange balls & partitions.

$\Rightarrow {}^{(n+r-1)}C_{(r-1)}$

Binomial Theorem for (−ve) Powers & Fractional Powers

1) Coeff. of t^r $\{r \in \mathbb{Z}_+\}$ in $(1-t)^{-n}$ $\{n \in \mathbb{N}\}$ $= {}^{(n+r-1)}C_r$

2) Coeff. of t^r $\{r \in \mathbb{Z}\}$ in $(1+t)^n$ is $\dfrac{n(n-1)(n-2)\dots(n-r+1)}{|\underline{r}}$

where $n \in \mathbb{R}$.